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The response of Hill's spherical vortex to a small axisymmetric disturbance

By H. K. MOFFATT
School of Mathematics, University of Bristol, England
AND D. W. MOORE
Department of Mathematics, Imperial College, London

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The response of a Hill's spherical vortex to an irrotational axisymmetric small perturbation is examined on the assumption that viscous diffusion of vorticity is negligible. The problem of determining the response is first reduced to a system of differential equations for the evolution of the Legendre coefficients of the disturbance stream function. This system is then solved approximately and it is shown that, if the initial disturbance is such as to make the vortex prolate spheroidal, the vortex detrains a fraction $\frac{3}{5}\epsilon$ of its original volume, where ϵ is the fractional extension of the axis of symmetry in the imposed distortion. The detrained fluid forms a thin spike growing from the rear stagnation point. If the vortex is initially oblate, irrotational fluid is entrained at the rear stagnation point to the interior of the vortex.

1. Introduction

Hill's spherical vortex (Hill 1894) provides one of the best-known examples of a steady rotational solution of the classical equations of inviscid incompressible fluid flow, and it is rather remarkable that its stability characteristics have not been investigated in detail.† The spherical vortex is known to be an extreme member of a one-parameter family of steady vortex rings (Norbury 1973) whose global stability to axisymmetric disturbances has been inferred by Benjamin (1976) in an approach to the stability problem based on the methods of functional analysis. Benjamin shows essentially that, subject to certain subsidiary conditions, the kinetic energy associated with any steady vortex ring is greater than the kinetic energy associated with any neighbouring axisymmetric unsteady solution of the governing equations; this property guarantees that disturbance energy cannot spontaneously grow at the expense of the parent vortex. The analysis below, based on linear perturbation theory, is consistent with this conclusion, but at the same time reveals a type of behaviour that lies outside the scope of the functional-analytic approach, viz. if the spherical surface of the Hill's vortex is subjected to an axisymmetric perturbation, then this perturbation decays exponentially at all points except in an exponentially decreasing neighbourhood of the rear stagnation point of the vortex (in a frame of reference in

† During the course of this work, it was drawn to our attention that Bliss (1973) has studied the problem by methods similar to those described here, and has obtained equations equivalent to (2.19) below. He did not attempt an analytical solution of these equations, but drew similar conclusions to ours, from a numerical study, concerning the failure of a truncation technique.

which the undisturbed flow is steady); in this neighbourhood, the perturbation *increases* without limit!

Physically, it is quite easy to see what is happening: since the vortex lines are frozen in the fluid, the surface of the vortex behaves like a material surface. The irrotational flow outside the vortex tends to sweep the perturbation towards the rear stagnation point where it develops an increasingly spiky structure. This sweeping process is modified by the self-induced velocity associated with the vorticity perturbation at the vortex surface; but the modification is not such as to prevent concentration and amplification of the disturbance at the rear stagnation point.

This type of spatially non-uniform behaviour is evidently associated with the presence in the undisturbed flow of a rear stagnation point. It seems most unlikely that similar behaviour could occur in the case of other (non-degenerate) members of the one-parameter family of vortex rings mentioned above. The process does nevertheless have some points of contact with the processes of entrainment and detrainment of vorticity in real ring vortices, as described by Maxworthy (1972) on the basis of visual studies. Maxworthy found that vorticity is in general spread throughout the whole volume of fluid that is carried along with a conventional vortex ring (although it is significantly weaker outside the vortex ring than inside). Viscous diffusion plays a dual role, in part causing a spread of vorticity and so an increase in the volume of the translating mass of fluid, and in part permitting detrainment of rotational fluid into a thin wake which emanates from the rear stagnation point. We shall find that, for Hill's vortex, both entrainment and detrainment processes can in fact be understood within the context of a purely *inviscid* analysis.

The rapid growth of the spike at the rear stagnation point of Hill's vortex is similar to the rapid growth of the viscous boundary layer at the rear stagnation point of an impulsively started cylinder (Proudman & Johnson 1962). In both cases, vorticity is found at small times in a region of outwardly convecting flow (viscous diffusion being the agency responsible for the initial generation of vorticity in the problem studied by Proudman & Johnson).

An axisymmetric perturbation of a Hill's vortex may be conceived in terms of an externally imposed irrotational distortion. For example, if a small Hill's vortex is embedded in irrotational flow through a contracting duct, then, relative to axes moving with the vortex, it experiences an axisymmetric irrotational strain, tending to make it prolate, with major axis along the axis of symmetry. Alternatively, if the small vortex is carried through a ring vortex with the same axis of symmetry it will experience first an extensive strain (tending to make it prolate), then a compressive strain (tending to restore the spherical shape); nonlinear evolution of the Hill's vortex during the process of interaction will presumably lead to a net residual axisymmetric perturbation which will continue to evolve after the actual interaction becomes negligible.

The particular property of the Hill's vortex that makes a stability analysis tractable is the uniformity of ω/s throughout the vortex, where ω is the azimuthal component of vorticity, and s represents distance from the axis of symmetry. This property persists under unsteady irrotational perturbations, since the vorticity equation in an inviscid incompressible fluid may be written in the simple form

$$D(\omega/s)/Dt = 0, \tag{1.1}$$

where D/Dt is the Lagrangian (or material) derivative. We shall restrict attention to externally imposed irrotational disturbances of the kind discussed above, so that, if $\omega = \lambda s$ in the undisturbed vortex, where λ is a constant, then $\omega = \lambda s$ in the disturbed vortex also. Attention may then be focused on the evolution of the bounding surface $S(t)$ of the region of rotational flow.

The equations governing small axisymmetric perturbations are derived in § 2. They can be reduced to an infinite set of ordinary differential equations for the coefficients in a Legendre expansion of the perturbation. This infinite system when truncated can be integrated without difficulty, but the truncation is valid only for comparatively small times. For longer times, the coefficients fail to decay as their order increases. An approximate treatment, valid for large times, is presented in § 3.

2. The evolution equations

The spherical vortex may be represented by the Stokes stream function

$$\psi_0(r,\theta) = \begin{cases} \tfrac{1}{2}U(1-a^3/r^3)\,r^2\sin^2\theta = \psi_{0+}(r,\theta) & (r>a) \\ -\tfrac{3}{4}U(1-r^2/a^2)\,r^2\sin^2\theta = \psi_{0-}(r,\theta) & (r<a) \end{cases} \tag{2.1}$$

in spherical polar co-ordinates (r,θ,χ) relative to an origin O at the centre of the vortex, with $\theta = 0$ parallel to the uniform stream U at infinity. Note that, with this orientation of axes, the rear stagnation point is at $r = a$, $\theta = 0$. The distance from the axis of symmetry is $s = r\sin\theta$. The vorticity distribution is $\boldsymbol{\omega} = \nabla \wedge \mathbf{u} = (0, 0, \omega_0)$, where

$$\omega_0 = \frac{-1}{r\sin\theta}D^2\psi_0, \quad D^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}, \tag{2.2}$$

i.e.

$$\omega_0 = \begin{cases} 0 & (r>a) \\ \lambda s & (r<a) \end{cases} \tag{2.3}$$

with $\lambda = -15U/2a^2$.

Suppose now that a small axisymmetric perturbation of the surface $r = a$ of the vortex is imposed at some initial instant $t = 0$. Then its surface $S(t)$ will subsequently distort according to an equation of the form

$$r = a(1+\epsilon h(\theta,t)). \tag{2.4}$$

Provided the perturbation is irrotational in the sense indicated in § 1, the stream function $\psi(r,\theta)$ of the disturbed flow must satisfy

$$D^2\psi = -r\sin\theta\,\omega(r,\theta) = \begin{cases} -\lambda r^2\sin^2\theta & \text{in } V_+ \\ 0 & \text{in } V_- \end{cases} \tag{2.5}$$

where $V_\pm$ denote the exterior and interior of $S(t)$ respectively. Let

$$\psi(r,\theta,t) = \begin{cases} \psi_{0+}(r,\theta) + \epsilon\psi_{1+}(r,\theta,t) & \text{in } V_+, \\ \psi_{0-}(r,\theta) + \epsilon\psi_{1-}(r,\theta,t) & \text{in } V_-. \end{cases} \tag{2.6}$$

Then, by virtue of (2.3) and (2.5), we have evidently

$$D^2\psi_{1-} = D^2\psi_{1+} = 0, \tag{2.7}$$

i.e. the disturbance stream function ψ_1 represents an irrotational flow.

The general axisymmetric solutions of (2.7), with ψ_{1-} regular at $r = 0$ and ψ_{1+} giving at most a dipole velocity field as $r \to \infty$, are

$$\left.\begin{aligned}\psi_{1-}(r,\theta,t) &= a^2U\sum_{n=1}^{\infty} A_n(t)\left(\frac{r}{a}\right)^{n+1}\sin^2\theta P_n'(\mu),\\ \psi_{1+}(r,\theta,t) &= a^2U\sum_{n=1}^{\infty} B_n(t)\left(\frac{a}{r}\right)^{n}\sin^2\theta P_n'(\mu),\end{aligned}\right\} \tag{2.8}$$

where $\mu = \cos\theta$.

The velocity field is $(u, v, 0)$, where

$$u = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}, \quad v = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}, \tag{2.9}$$

and is thus given by

$$\left.\begin{aligned}u &= -\tfrac{3}{2}U\left(1-\frac{r^2}{a^2}\right)\cos\theta + \epsilon U\sum_1^{\infty} A_n\left(\frac{r}{a}\right)^{n-1} n(n+1)P_n(\mu)\\ v &= \tfrac{3}{2}U\left(1-\frac{2r^2}{a^2}\right)\sin\theta - \epsilon U\sum_1^{\infty} A_n\left(\frac{r}{a}\right)^{n-1}(n+1)\sin\theta P_n'(\mu)\end{aligned}\right\} \text{ in } V_-, \tag{2.10}$$

$$\left.\begin{aligned}u &= U\left(1-\frac{a^3}{r^3}\right)\cos\theta + \epsilon U\sum_1^{\infty} B_n\left(\frac{a}{r}\right)^{n+2} n(n+1)P_n(\mu)\\ v &= -U\left(1+\frac{a^3}{2r^3}\right)\sin\theta + \epsilon U\sum_1^{\infty} B_n\left(\frac{a}{r}\right)^{n+2} n\sin\theta P_n'(\mu)\end{aligned}\right\} \text{ in } V_+. \tag{2.11}$$

Continuity of u (or of ψ) across $S(t)$ to order ϵ gives

$$A_n = B_n. \tag{2.12}$$

Continuity of v (or equivalently of the pressure field) across $S(t)$ to order ϵ then gives

$$\sum_{n=1}^{\infty}(2n+1)A_n(t)P_n'(\mu) = -\frac{15}{2}h(\mu,t). \tag{2.13}$$

Now the surface $S(t)$ is a material surface in the fluid; hence

$$\frac{D}{Dt}(r-a-a\epsilon h(\theta,t)) = 0,$$

or equivalently

$$u = a\epsilon\left(\frac{\partial h}{\partial t}+\frac{v}{r}\frac{\partial h}{\partial\theta}\right) \quad \text{on} \quad S(t). \tag{2.14}$$

Hence, correct to order ϵ, we have, using (2.4) and (2.10),

$$\frac{a}{U}\frac{\partial h}{\partial t} = -\frac{3}{2}\frac{\partial}{\partial\mu}[(1-\mu^2)h] + \sum_{n=1}^{\infty}A_n n(n+1)P_n(\mu), \tag{2.15}$$

or, using the expansion (2.13), and the Legendre equation satisfied by $P_n(\mu)$,

$$\frac{a}{U}\frac{\partial h}{\partial t} = -\frac{1}{5}\sum_1^{\infty}(2n+1)n(n+1)A_nP_n(\mu) + \sum_{n=1}^{\infty}A_n n(n+1)P_n(\mu) \tag{2.16}$$

$$= -\frac{2}{5}\sum_1^{\infty}(n-2)n(n+1)A_nP_n(\mu). \tag{2.17}$$

The first term on the right of (2.15) [or (2.16)] represents convection of the surface of the vortex by the velocity field associated with the undisturbed flow, while the second term represents the effect of the induced velocity at the unperturbed surface $r = a$ due to the vorticity perturbation distributed over the whole surface. It is evident from comparison of the two series on the right of (2.16) that the former effect dominates when $n \gg 1$.

From (2.13), (2.17), and the identity

$$(2n+1)P_n(\mu) = P'_{n+1}(\mu) - P'_{n-1}(\mu) \quad (n = 1, 2, \ldots), \tag{2.18}$$

we now easily find the equations satisfied by the coefficients $A_n(t)$, viz.

$$(2n+1)\frac{a}{U}\frac{dA_n}{dt} = 3(n-1)\left[\frac{n(n-3)}{2n-1}A_{n-1} - \frac{(n+1)(n+2)}{2n+3}A_{n+1}\right] \quad (n = 1, 2, \ldots). \tag{2.19}$$

Note immediately that $dA_1/dt = 0$, so that

$$A_1 = \text{constant} = C \quad \text{say}. \tag{2.20}$$

This merely reflects the conservation of impulse for the disturbed vortex. The impulse I and kinetic energy T of the vortex (Batchelor 1967, chap. VII) are in fact given by

$$I/\rho U a^3 = -2\pi + 4\pi\epsilon A_1 + O(\epsilon^2), \tag{2.21}$$

$$T/\rho U^2 a^3 = \tfrac{10}{7}\pi - 4\pi\epsilon A_1 + O(\epsilon^2), \tag{2.22}$$

and constancy of both quantities (at order ϵ) is guaranteed by (2.20).

The coefficients $A_n(t)$ are simply related to the coefficients $h_n(t)$ defined by

$$h(\theta, t) = \sum_0^\infty h_n(t)P_n(\mu). \tag{2.23}$$

Comparing (2.23) and (2.13), and using (2.18) again, we have

$$\frac{h_{n-1}}{2n-1} - \frac{h_{n+1}}{2n+3} = -\frac{2}{15}(2n+1)A_n \quad (n = 1, 2, 3, \ldots). \tag{2.24}$$

Summing the odd and even members of this set of equations gives

$$h_0 = -\frac{2}{15}\sum_1^\infty (4n-1)A_{2n-1}, \tag{2.25}$$

$$\tfrac{1}{3}h_1 = -\frac{2}{15}\sum_1^\infty (4n+1)A_{2n}. \tag{2.26}$$

Now conservation of the volume of rotational fluid implies that $h_0 = 0$ for all t; in fact, inspection of the system (2.19) does confirm that

$$\frac{d}{dt}\sum_1^\infty (4n-1)A_{2n-1}(t) = 0, \tag{2.27}$$

consistent with (2.25). We also obtain

$$\frac{d}{dt}\sum_{n=1}^\infty (4n+1)A_{2n} = \frac{2U}{a}A_1 = \frac{2U}{a}C, \tag{2.28}$$

so that, from (2.26),

$$h_1 = -4CUt/5a + D, \tag{2.29}$$

where D is a constant of integration; we may choose the time origin $t = 0$ so that $D = 0$. It is evident that (2.29) describes a translation of the vortex without distortion relative to the origin in which the unperturbed vortex is stationary.

We expect that, so long as the surface of the vortex does not exhibit any large gradients, the coefficients A_n will decrease rapidly with n. This suggests solving the system (2.19) by setting $A_{N+1}, A_{N+2}, \ldots$ equal to zero for some value of N. (This was the approach adopted by Bliss 1973.)

The equations were integrated (with $N = 50$) using a fourth-order Runge–Kutta scheme, the choice of time step being dictated by the shortest time scale occurring in the solution; this decreases with N in a way we estimate theoretically from the results of § 3. The initial disturbance is a spheroidal distortion, i.e.

$$\left.\begin{aligned} h_n(0) &= 0 \quad (n \neq 2), \\ h_2(0) &= 1; \end{aligned}\right\} \tag{2.30}$$

the values of $A_n(0)$ are obtained from (2.24) and the results are shown in table 1. Clearly when $Ut/a = 3{\cdot}0$, the coefficients are not declining rapidly enough with n to justify the truncation.

The reason for this behaviour can be deduced from an approximate solution of the governing equations (2.19) valid for $n \gg 1$. As a first step, the equations (2.19) are expressed in a simpler form in terms of new variables. Let

$$\alpha_n = \frac{n(n+2)(n+3)}{2n+5} A_{n+2} \quad (n = 0, 1, 2, \ldots) \tag{2.31}$$

and

$$\tau = 3Ut/4a. \tag{2.32}$$

The equations (2.19) then become

$$d\alpha_n/d\tau = c_n(\alpha_{n-1} - \alpha_{n+1}) \quad (n = 1, 2, \ldots) \tag{2.33}$$

where

$$c_n = n[1-(2n+5)^{-2}]. \tag{2.34}$$

When $\alpha_1, \alpha_2, \alpha_3, \ldots$ have been determined by solving this system of equations, the values of $A_3, A_4, A_5, \ldots$ are determined at once. Then A_1 and A_2 are given by

$$A_1 = C, \quad \frac{5a}{U}\frac{dA_2}{dt} = -2A_1 - \tfrac{36}{7}A_3. \tag{2.35}$$

The character of the solutions of (2.33) may be anticipated by the following argument. For $n \gg 1$, $c_n \sim n$, and (2.33) may be compared with the partial differential equation

$$\partial\alpha/\partial\tau = -2n\,\partial\alpha/\partial n \tag{2.36}$$

for a function $\alpha(n, \tau)$, n being regarded as a continuous variable. The solution of (2.36) satisfying an initial condition $\alpha(n, 0) = H(n)$ is

$$\alpha(n, \tau) = H(ne^{-2\tau}). \tag{2.37}$$

For example, if

$$H(n) = \begin{cases} 1 & (n_1 \leqslant n \leqslant n_2), \\ 0 & \text{elsewhere}, \end{cases} \tag{2.38}$$

then

$$\alpha(n, \tau) = \begin{cases} 1 & (n_1 e^{2\tau} \leqslant n \leqslant n_2 e^{2\tau}), \\ 0 & \text{elsewhere}. \end{cases} \tag{2.39}$$

Ut/a	A_{10}	A_{20}	A_{30}	A_{40}	A_{50}
1·0	$-1{\cdot}698\times10^{-3}$	$-9{\cdot}144\times10^{-6}$	$-6{\cdot}526\times10^{-8}$	$-5{\cdot}233\times10^{-10}$	$-6{\cdot}332\times10^{-12}$
2·0	$-6{\cdot}352\times10^{-3}$	$-1{\cdot}174\times10^{-3}$	$-2{\cdot}910\times10^{-4}$	$-9{\cdot}026\times10^{-5}$	$-4{\cdot}345\times10^{-5}$
3·0	$-2{\cdot}894\times10^{-3}$	$-1{\cdot}691\times10^{-3}$	$-1{\cdot}063\times10^{-3}$	$-6{\cdot}752\times10^{-4}$	$-4{\cdot}437\times10^{-4}$
4·0	$-6{\cdot}605\times10^{-3}$	$-1{\cdot}729\times10^{-3}$	$-7{\cdot}121\times10^{-4}$	$-3{\cdot}836\times10^{-4}$	$-2{\cdot}414\times10^{-4}$

TABLE 1. The Legendre coefficients of the stream function obtained from the governing equations (2.19) truncated at A_{50}.

This indicates the manner in which the 'spectrum' of the disturbance evolves towards higher values of n as τ increases.

3. The behaviour for large τ

It is evident from (2.34) that replacement of c_n by n is a good approximation for all $n \geqslant 1$; the error when $n = 1$ is only 2% and it decreases rapidly as n increases. We therefore study here the behaviour of the system

$$d\alpha_n/d\tau = n(\alpha_{n-1}-\alpha_{n+1}) \quad (n = 1, 2, \ldots), \tag{3.1}$$

where

$$\alpha_0(\tau) = 0. \tag{3.2}$$

The initial values of α_n are given in terms of the initial distortion by

$$\alpha_n(0) = -\frac{15n}{8}\left(1-\frac{1}{(2n+5)^2}\right)\left(\frac{h_{n+1}(0)}{2n+3}-\frac{h_{n+3}(0)}{2n+7}\right) \quad (n = 1, 2, \ldots), \tag{3.3}$$

or, consistent with the approximation leading to (3.1),

$$\alpha_n(0) = -\frac{15n}{8}\left(\frac{h_{n+1}(0)}{2n+3}-\frac{h_{n+3}(0)}{2n+7}\right) \quad (n = 1, 2, \ldots). \tag{3.4}$$

The system (3.1) admits a particular solution

$$\alpha_n^0(\tau) = (\operatorname{th}\tau)^n \quad (n = 0, 1, 2, \ldots) \tag{3.5}$$

(where $\operatorname{th}\tau \equiv \tanh\tau$); from this we can construct a family of particular solutions

$$\alpha_n^{(r)}(\tau) = \frac{1}{r!}\left(\frac{d}{d\tau}\right)^r (\operatorname{th}\tau)^n, \tag{3.6}$$

each of which satisfies (3.2) and the initial conditions

$$\alpha_n^{(r)}(0) = \begin{cases} 1 & (n = r), \\ 0 & (n > r). \end{cases} \tag{3.7}$$

Thus if $\alpha_n(0) = 0$ for all $n > n_0$ we can construct a solution of (3.1) and (3.2) by forming a linear combination

$$\alpha_n(\tau) = \sum_{r=1}^{n_0} \beta_r \alpha_n^{(r)}(\tau); \tag{3.8}$$

it is easy to see that the constants β_r are uniquely determined.

As in the numerical work described in § 2, we select for detailed study the case of an initially spheroidal distortion, so that the constants $h_n(0)$ have the values specified in (2.30). It follows from (3.4) that

$$\left.\begin{aligned}\alpha_1(0) &= -\tfrac{3}{8},\\ \alpha_n(0) &= 0 \quad (n \geqslant 2).\end{aligned}\right\} \tag{3.9}$$

This is the case $n_0 = 1$ and thus

$$\alpha_n(\tau) = -\frac{3}{8}\frac{d}{d\tau}(\operatorname{th}\tau)^n \quad (n = 0, 1, 2, \ldots). \tag{3.10}$$

It is easy to verify that, for large n, $|\alpha_n(\tau)|$ has a maximum when

$$\tau = \cosh^{-1}\left(\frac{n+1}{2}\right)^{\frac{1}{2}} = \tfrac{1}{2}\log(2n) + O\left(\frac{1}{n^2}\right), \tag{3.11}$$

and the maximum value itself is

$$|\alpha_n(\tau)|_{\max} = \frac{3n}{4n+1}\left(\frac{n-1}{n+1}\right)^{\frac{1}{2}(n-1)} \sim \tfrac{3}{4}e^{-1}. \tag{3.12}$$

This confirms the predicted shift of the spectrum towards large values of n as time increases. Note that (3.11) is compatible with the approximation embodied in (2.36), which implies that $\alpha(n,\tau)$ is constant on the characteristic curves $2\tau = \log n + \text{constant}$.

The initial conditions (2.30) and the relation (2.24) give $A_1(0) = \frac{1}{2}$ and hence, from (2.35), $A_1(\tau) = \frac{1}{2}$. Thus, from (2.13) and (2.31)

$$h = -\tfrac{1}{5} - 2A_2(t)\,\mu + \tfrac{1}{20}\operatorname{sech}^2\tau \sum_{n=3}^{\infty} \frac{(2n+1)^2}{n(n+1)}(\operatorname{th}\tau)^{n-3}P_n'(\mu). \tag{3.13}$$

Now

$$\frac{(2n+1)^2}{n(n+1)} = 4\left(1 + \frac{1}{4n(n+1)}\right), \tag{3.14}$$

so that it is consistent with the approximations already made in obtaining (3.1) and (3.4) to replace the left-hand side of (3.14) by 4 in working out the sum in (3.13). Thus we find

$$h = -\tfrac{1}{5} - 2A_2(t)\,\mu + \frac{\operatorname{cosech}^2\tau}{5\operatorname{th}\tau}\frac{d}{d\mu}\sum_3^{\infty}(\operatorname{th}\tau)^n P_n(\mu). \tag{3.15}$$

The infinite series is easily summed and finally

$$h = -\tfrac{1}{5} - 2A_2(t)\,\mu + \tfrac{1}{5}\operatorname{cosech}^2\tau[(1 - 2\mu\operatorname{th}\tau + \operatorname{th}^2\tau)^{-\frac{3}{2}} - 1 - 3\mu\operatorname{th}\tau]. \tag{3.16}$$

The asymptotic behaviour for $\tau \gg 1$ is now easily determined. With the approximations

$$\operatorname{cosech}^2\tau \sim 4e^{-2\tau} \quad \text{and} \quad \operatorname{th}\tau \sim 1 - 2e^{-2\tau},$$

(3.16) gives

$$h \sim -\tfrac{1}{5} + (-2A_2(\tau) - \tfrac{12}{5}e^{-2\tau})\,\mu + \tfrac{4}{5}e^{-2\tau}\{[4e^{-4\tau} + 2(1-\mu)]^{-\frac{3}{2}} - 1\}, \tag{3.17}$$

an expression that is uniformly valid in μ. The singular behaviour near the rear stagnation point $\mu = 1$ is now readily apparent. If μ is fixed and not equal to 1 then

$$h \sim -\tfrac{1}{5} - \tfrac{8}{15}\mu(\tau + \text{constant}) + O(e^{-2\tau}), \tag{3.18}$$

using (2.35) to determine the asymptotic behaviour of A_2. The term proportional to μ represents a steady drift without change of shape; the propagation speed is altered by a factor $1-\frac{2}{5}\epsilon$. Thus, apart from a small residual change of radius, the perturbation dies away at every point of the surface except in a decreasing neighbourhood of the rear stagnation point. Right at the rear stagnation point, where $\mu = 1$, (3.17) gives

$$h \sim \tfrac{1}{10}e^{4\tau}, \tag{3.19}$$

so that the perturbation here increases exponentially. Thus we can form the following picture of the development of the perturbation. When $\epsilon > 0$ (the perturbed vortex being then slightly prolate) vorticity is swept off the surface $r = a$ (leaving it at the slightly diminished radius $a(1-\frac{1}{5}\epsilon)$) and is shed into a spike in the neighbourhood $\mu \approx 1$. When $\epsilon < 0$ (corresponding to an initially oblate perturbation) the radius *increases* to $a(1-\frac{1}{5}\epsilon)$, this increase being brought about by entrainment of irrotational fluid into the region $r < a$ in the neighbourhood $\mu \approx 1$. The change of propagation speed is just that required by this change of radius.

The detailed shape of the spike can be found from (3.17); restoring the physical variables r, θ and t,

$$r = a+\tfrac{1}{10}a\epsilon\exp(3Ut/a)\,[1+\tfrac{1}{4}\theta^2\exp(3Ut/a)]^{-\frac{3}{2}}. \tag{3.20}$$

According to (3.20) the volume of the spike is V, where

$$V = \tfrac{1}{5}\pi a^3\epsilon\exp(3Ut/a)\int_0^{\hat{\theta}}\theta[1+\tfrac{1}{4}\theta^2\exp(3Ut/a)]^{-\frac{3}{2}}\,d\theta, \tag{3.21}$$

where $\hat{\theta}$ is any angle such that

$$\exp(-3Ut/2a) \ll \hat{\theta} \ll 1. \tag{3.22}$$

The integral is easily evaluated to give (for sufficiently large t),

$$V \sim \tfrac{4}{5}\pi\epsilon a^3, \tag{3.23}$$

which is just the volume lost by the Hill's vortex when its radius shrinks from a to $a(1-\frac{1}{5}\epsilon)$. Thus, when $\epsilon > 0$, a definite volume of rotational fluid is detrained from the Hill's vortex into a spike growing from the rear stagnation point. Similarly, when $\epsilon < 0$, a definite volume of irrotational fluid is entrained in an indented spike at the rear stagnation point. The form of (3.20) reveals that the growth of the spike in either case is controlled entirely by the convective effects of the undisturbed flow field, as anticipated in §2. The motion of a fluid particle near the rear stagnation point in the flow field given by (2.1) is governed by the equations

$$a\,dr/dt = 3U(r-a), \quad a\,d\theta/dt = -\tfrac{3}{2}U\theta, \tag{3.24}$$

which may be integrated to give

$$\left.\begin{aligned} r-a &= (r_0-a)\exp[3Ua^{-1}(t-t_0)], \\ \theta &= \theta_0\exp[-3U(t-t_0)/2a], \end{aligned}\right\} \tag{3.25}$$

where r_0 and θ_0 are the co-ordinates of the particle at $t = t_0$. If we use the solution (3.25) to determine the motion of fluid particles which lie at $t = t_0$ on the surface

$$r_0-a = \tfrac{1}{10}a\epsilon\exp(3Ut_0/a)\,[1+\tfrac{1}{4}\theta_0^2\exp(3Ut_0/a)]^{-\frac{3}{2}}, \tag{3.26}$$

we see at once that at time t ($>t_0$) they lie on the surface (3.20).

This suggests a way of removing the restriction to small disturbances. Equation (3.20) is valid so long as

$$\tfrac{1}{10}\epsilon \exp(3Ut/a) \ll 1, \qquad (3.27)$$

that is, so long as the spike is small. Pick a time t_0 for which (3.27) holds and follow the subsequent evolution of the surface (3.26) using the full equations for convection of particles near the axis. When $\epsilon > 0$, so that the spike grows outwards, these are

$$\left.\begin{aligned} dr/dt &= U(1-a^3/r^3), \\ r d\theta/dt &= -U(1+a^3/2r^3), \end{aligned}\right\} \qquad (3.28)$$

and when $\epsilon < 0$, so that the spike grows inwards,

$$\left.\begin{aligned} dr/dt &= -\tfrac{3}{2}U(1-r^2/a^2), \\ r d\theta/dt &= \tfrac{3}{2}U(1-2r^2/a^2)\,\theta. \end{aligned}\right\} \qquad (3.29)$$

Results obtained in this way for $\epsilon = +10^{-4}$ are shown in figure 1. Evidently the spike grows very rapidly even for a small initial disturbance. This conclusion is supported by calculating the time needed for the spike to reach a point $2a$ downstream of the rear stagnation point.

The calculation is in two stages. First, according to linear theory the fluid particle initially at $r = a(1+\epsilon)$, $\theta = 0$ is at $r = a(1+\eta)$, $\theta = 0$ at $t = t_0$, where, according to (3.19), if $t_0 \gg a/U$,

$$\eta = \tfrac{1}{10}\epsilon \exp(3Ut_0/a), \qquad (3.30)$$

and this is valid so long as $\eta \ll 1$. Secondly, the time T at which a fluid particle starting at $r = a(1+\eta)$ at $t = t_0$ arrives at $r = 2a$ can be found from the first of equations (3.28) and satisfies

$$U(T-t_0)/a = 1{\cdot}824\ldots + \tfrac{1}{3}\log(\eta^{-1}) + O(\eta). \qquad (3.31)$$

Thus on substituting for η from (3.30) we find that t_0 cancels and

$$UT/a = 2{\cdot}592\ldots + \tfrac{1}{3}\log(\epsilon^{-1}) + O(\epsilon). \qquad (3.32)$$

When $\epsilon = 10^{-4}$, $UT/a = 5{\cdot}66\ldots$, so that the spike is one diameter downstream by the time the vortex has travelled about $2\frac{1}{2}$ diameters.

When $\epsilon < 0$, the indented spike will approach the *forward* stagnation point, although, under the effect of convection alone, it will never quite reach it; its surface will instead presumably be increasingly distorted by the internal convective action of the vortex, its meridian section ultimately developing a tight double spiral structure (see, for example, figure 12 of Maxworthy 1972). Viscosity would of course in time tend to eliminate variations in ω/s within the new vortex.

A surprising feature of the results presented in this section is that the response of Hill's vortex contains no oscillatory components. This conclusion is, however, based on the approximate treatment of the differential equations embodied in (3.1) and (3.4). This approximation can be shown not to violate conservation of volume or momentum and to lead to a set of coefficients A_n which differ from the values obtained by truncation by only 5–10 %; of course, the time at which the comparison is made must be small enough for the truncation method to be valid.

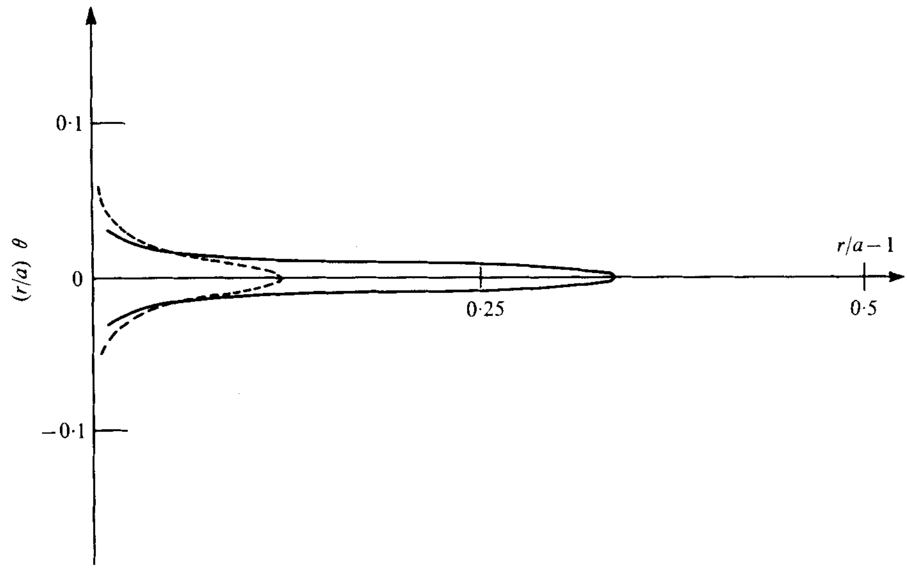

FIGURE 1. The shape of the spike in the case $\epsilon = 10^{-4}$ for $Ut/a = 3{\cdot}2$ (dashed curve) and $Ut/a = 3{\cdot}7$ (continuous curve).

However, it is worthwhile to examine the exact system (2.19) to see whether or not it permits free oscillations. If we write

$$A_n(t) = \bar{A}_n \exp(i\omega Ut/a), \tag{3.33}$$

and substitute in the differential equations (2.19), we find that

$$(2n+1)\, i\omega \bar{A}_n = 3(n-1)\left[\frac{n(n-3)}{2n-1}\bar{A}_{n-1} - \frac{(n+1)(n+2)}{2n+3}\bar{A}_{n+1}\right] \quad (n = 1, 2, 3, \ldots). \tag{3.34}$$

The asymptotic form of the solution of this system of difference equations for large n is easily determined:

$$\bar{A}_n \sim Cn^{-\frac{2}{3}i\omega} + D(-1)^n n^{\frac{2}{3}i\omega}, \tag{3.35}$$

where C and D are constants. An acceptable solution must satisfy $\bar{A}_n \to 0$ at an exponential rate as $n \to \infty$ and this behaviour cannot be secured by any real choice of ω. Thus free oscillations are not possible.

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